Supplementary Material for: Efficient multiphoton generation in waveguide QED.

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In this Supplementary Material, we provide the full details of different protocols described within the main manuscript. For the sake of a better understanding, we make the Sections for each protocol self-contained, discussing for each of them the required atom-waveguide tools and the calculation of probabilities and fidelities. The outline of the Supplementary Material reads as follows:

- In Section SM1 we give the common guidelines to understand the analysis of the different protocols.
- In Section SM2, we discuss the first protocol of the main manuscript which heralds the generation of m collective excitations using single photon detections within the waveguide. We discuss both the situation where we accumulate directly the excitations into the same atomic level and where we store it in another hyperfine level and combine them a posteriori.
- In Section SM3, we discuss the second protocol of the main manuscript which accumulates excitations directly within the same level using a two-step protocol. The protocol overcomes the trade-off between probability of success and fidelities of the first protocol but still requires an exponential number of operations.
- In Section SM4, we discuss the third protocol which shows how to overcome the exponential scaling of the number of operations of the second protocol by using additional auxiliary levels to store the excitations after each heralded transfer while keeping the overall infidelity still low.
- Finally, in Section SM5 we discuss how to use beam splitter transformations and atomic detection to merge atomic excitations and reach high photon numbers.

SM1. GENERAL GUIDELINES TO ANALYZE THE PROTOCOLS.

In all the protocols the goal is the generation of m collective excitations in a collection of N atoms that we name the *target* ensemble. We denote such state as $|\Psi_m\rangle$. Moreover, our protocols rely on heralding the excitations one by one by measuring a given state in the auxiliary system that we denote by $|\Phi_{\text{her}}\rangle$, which is either directly the photonic state within the waveguide (first protocol) or an auxiliary atomic state (i.e., the detector ensemble in the second and third protocols). Therefore, in order to analyze the protocols we study two things:

- First, we analyze the scaling of the probability and the fidelity of heralding the addition of a single excitation. This means that we start with a density matrix $\rho_m(0) = |\Psi_m\rangle \otimes |\Phi_0\rangle\langle\Psi_m| \otimes \langle\Phi_0|$, where $|\Phi_0\rangle$ is the initial state of the auxiliary system. After a given evolution of the density matrix, denoted by $S\rho_m(0)$, we calculate the probability of succeeding in the heralding which reads

$$p = \text{Tr}\left[\langle\Phi_{\text{her}}|S\rho_m(0)|\Phi_{\text{her}}\rangle\right]. \tag{SM1}$$

  However, the heralding is in general not perfect and we might have created states in the target ensemble other than the desired one, i.e., $|\Psi_{m+1}\rangle$. This is quantified by the fidelity after heralding, which reads

$$F_{m\to m+1} = \frac{\langle\Psi_{m+1}|\langle\Phi_{\text{her}}|S\rho_m(0)|\Phi_{\text{her}}\rangle|\Psi_{m+1}\rangle}{p} \tag{SM2}$$

  The infidelity is defined as: $I_{m\to m+1} = 1 - F_{m\to m+1}$.

- Once we have analyzed the previous general step, we consider the complete process to arrive from $|\Psi_0\rangle \to |\Psi_1\rangle \to \cdots \to |\Psi_m\rangle$ and calculate the average number of repetitions, R_m, to arrive to $|\Psi_m\rangle$ and the average infidelity of the process I_m. Both R_m and I_m depend on the way we merge the excitations and changes with the different protocols, as we show below.

SM2. HERALDING MULTIPLE COLLECTIVE ATOMIC EXCITATIONS IN WAVEGUIDE QED SETUPS USING SINGLE PHOTON DETECTORS.

The protocol consists in heralding the transfer of a single collective atomic excitations m times through m photon detection events within the waveguide. We discuss two different approaches: i) one where we store the heralded single excitation in a different level s_1 with methods that we revise in Section SM5; ii) or where we store the excitation directly into the same level.

The key step in both cases is to study how an additional single collective excitation can be added to an atomic ensemble which already contains m excitations in another level, s_1, i.e.

$$|\Phi_m^{s_1}\rangle = \frac{1}{\sqrt{\binom{N}{m}}}\text{sym}\{|s_1\rangle^{\otimes m}|g\rangle^{\otimes(N-m)}\}. \tag{SM3}$$

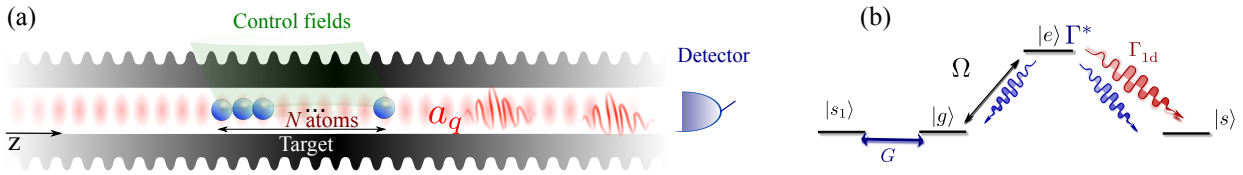

Figure SM1. (a) General setup for first protocol discussed in the main manuscript which consists of a collective ensemble where we prepare superpositions and a single photon detector that we use to herald the excitations. (b) Internal level structure of emitters in which the transition $e \leftrightarrow s$ is coupled to a waveguide mode with rate Γ_{1d}. The transition $g \leftrightarrow e$ is controlled through the Raman laser Ω. We also include the possibility of having an extra auxiliary level s_1 where excitations can be stored using microwave [or two-photon Raman] fields G.

A. Atom-waveguide resources

The tools/resources that we use are:

- N atoms trapped close to a one-dimensional waveguide as depicted in Fig. SM1 (a), with a level structure, as shown in Fig. SM1(b), where one optical transition $e \leftrightarrow s$ is coupled to a waveguide mode with a rate Γ_{1d}, and the other leg $e \to g$ is controlled by a collective classical field $\Omega^n = \Omega$. In one of the schemes that we explore, we also assume that we have an extra mode s_1, where the excitations can be stored and which can be combined through microwave or two-photon Raman transitions with Rabi frequency G.

- The atoms are placed at distances commensurate with the characteristic wavelength of the guided mode, λ_a, at the frequency of the transition $e \to g$, i.e. $z_n = n\lambda_a$. With that assumption, it is easy to show that effective atom dynamics induced by the interaction with the waveguide is solely driven by long-range dissipative couplings:

$$\mathscr{L}_{\text{coll}}[\rho] = \frac{\Gamma_{1d}}{2} D_{S_{ge}}[\rho], \tag{SM4}$$

  where ρ is the reduced density matrix of the atomic system, $D_O[\rho] = (2O\rho O^\dagger - O^\dagger O\rho - \rho O^\dagger O)$ is the dissipator associated to a given jump operator O, and where we defined the collective spin operator of the target ensemble as $S_{\alpha\beta} = \sum_{n=1}^{N} \sigma^n_{\alpha\beta}$, with $\sigma^n_{\alpha\beta} = |\alpha\rangle_n\langle\beta|$. Obviously, the excited state can also decay to other modes other than into the waveguide, which leads to extra Lindblad terms which read:

$$\mathscr{L}_*(\rho) = \sum_{n=1,\ldots,N} \left(\frac{\Gamma^*}{4} D_{\sigma^n_{se}}[\rho] + \frac{\Gamma^*}{4} D_{\sigma^n_{ge}}[\rho] \right), \tag{SM5}$$

  which lead to finite Purcell factor $P_{1d} = \frac{\Gamma_{1d}}{\Gamma^*}$.

- A single photon detector with overall efficiency η at the end of the waveguide to detect the emission of waveguide photons.

B. Protocol and calculation of probabilities.

The protocol works as follows: we apply a laser pulse during a short time T such that $\sqrt{N}\Omega T/2 = x \ll 1$. Then, the stored superposition $|\Phi^{s_1}_m\rangle$ evolves into:

$$|\Psi_0\rangle \propto |\Phi^{s_1}_m\rangle + x\frac{1}{\sqrt{N_m}} S_{eg}|\Phi^{s_1}_m\rangle + \frac{x^2}{2}\sqrt{\frac{2}{N_m(N_m-1)}} S^2_{eg}|\Phi^{s_1}_m\rangle + O(x^3). \tag{SM6}$$

Then, we let the system evolve only through the interaction with the waveguide and the bath of leaky photons. In order to obtain the probabilities associated to different process, we use the expansion of the master equation [1] distinguishing the evolution of the effective non-hermitian Hamiltonian, i.e., $S(t,t_0)\rho = e^{-iH_{\text{eff}}t}\rho e^{iH^\dagger_{\text{eff}}t}$, and the one resulting from quantum jumps evolution, i.e., $J\rho$, that is given in general by:

$$\rho(t) = S(t,t_0)\rho(t_0) + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_n S(t,t_n) J(t_n) \ldots \int_{t_0}^{t_2} dt_1 S(t_1,t_0)\rho(t_0). \tag{SM7}$$

The non-hermitian evolution leads to:

$$|\Psi(t)\rangle \propto e^{-iH_{\text{eff}}t}|\Psi_0\rangle \propto |\Phi_m^{s_1}\rangle + x\frac{e^{-(\Gamma_{1\text{d}}+\Gamma^*)t}}{\sqrt{N_m}}S_{eg}\,|\Phi_m^{s_1}\rangle + \frac{x^2}{2}e^{-2(\Gamma_{1\text{d}}+\Gamma^*)t}\sqrt{\frac{2}{N_m(N_m-1)}}S_{eg}^2|\Phi_m^{s_1}\rangle + O(x^3)\,, \tag{SM8}$$

where we see that if we wait a time long enough, i.e., $t \gg (\Gamma^*)^{-1}$ the only population remaining in $|\Psi(T)\rangle \to |\Phi_m^{s_1}\rangle$. The probability emitting a collective photon from $S_{eg}|\Phi_m^{s_1}\rangle$ is given by:

$$p_{\text{coll}} = \frac{x^2}{1+\frac{1}{P_{1\text{d}}}} \approx x^2\left(1-\frac{1}{P_{1\text{d}}}\right), \tag{SM9}$$

where in the last approximation we assumed to be in a regime with $P_{1\text{d}} \gg 1$, whereas the one of emitting an spontaneous emission photon from the same state:

$$p_* = \frac{x^2}{P_{1\text{d}}+1} \approx \frac{x^2}{P_{1\text{d}}}\,. \tag{SM10}$$

We herald with the detection of a waveguide photon with efficiency η, such that the final success probability of heralding is:

$$p = p_{\text{coll}} \times \eta \approx \eta x^2\left(1-\frac{1}{P_{1\text{d}}}\right), \tag{SM11}$$

In case of no detection it may have happened that our waveguide photon was emitted but not detected, that is, the atoms will indeed be in a superposition $S_{sg}\,|\Phi_m^{s_1}\rangle$. So, before making a new attempt, we ensure to pump back any possible excitation in g to s minimizing the emission into leaky modes, which can be done as follows:

- First, apply a π pulse with a microwave field that flips all the excitations $g \to s$. This switches $S_{sg}|\Phi_m^{s_1}\rangle$ into $S_{gs}|\Psi_m^{s_1}\rangle$, where $|\Psi_m^{s_1}\rangle$ is the same as $|\Phi_m^{s_1}\rangle$ but with levels s and g interchanged.
- Then, we apply a fast Raman π-pulse with $\Omega \gg N\Gamma_{1\text{d}}$ such that $S_{gs}|\Psi_m^{s_1}\rangle \to S_{es}|\Psi_m^{s_1}\rangle \to |\Psi_m^{s_1}\rangle$. This incoherent transfer is done through a collective photon, such that the probability of emitting a leaky photon is:

$$p_{\text{pump},*} \approx (1-\eta)\times p_{\text{coll}} \times \frac{1}{N_m P_{1\text{d}}} \approx (1-\eta)\frac{x^2}{N_m P_{1\text{d}}}\,. \tag{SM12}$$

- Finally, we reverse the microwave π-pulse such that $g \leftrightarrow s$ and therefore $|\Psi_m^{s_1}\rangle \to |\Phi_m^{s_1}\rangle$.

C. Fidelities of the protocol.

In order to calculate the errors (and fidelities) of the protocol we have to distinguish between the errors introduced after successfully heralding the excitations, and the error per trial that we introduce when we repeat the protocol after failure.

- *Successful heralding:* It can be shown that if we detect a single photon in the waveguide, this means that the initial density matrix $\rho_m = |\Psi_m\rangle\langle\Psi_m|$ transforms into:

$$\rho_m \to \frac{1}{N_m}S_{gs}\rho_m S_{sg} + \frac{2x^2(1-\eta_{\text{d}})}{N_m(N_m-1)}S_{gs}^2\rho_m S_{sg}^2\,, \tag{SM13}$$

  where the first term is the desired process, whereas the second corresponds to the probability of detecting only one of the two photons emitted from the doubly excited terms $S_{es}^2\,|\Phi_m^{s_1}\rangle$. This introduces a large error as the state $S_{gs}\,|\Phi_m^{s_1}\rangle$ is orthogonal to the state that we want to create, such that the error when heralding is:

$$\varepsilon_{\text{double}} = x^2(1-\eta)\,. \tag{SM14}$$

- *Failed heralding:* If we detect no photon, then, our state is projected to:

$$\rho_m \to (1-x^2)\rho_m + p_{\text{coll}}(1-\eta)\frac{1}{N_m}S_{gs}\rho_m S_{sg} + p_*\mathbb{J}_*\rho_m + O(x^4)\,, \tag{SM15}$$

which corresponds to processes in which: i) we have not created any excitation in the system, with probability $(1-x^2)$, ii) we have created a single collective excitation emitting a collective photon but we have not detected it; iii) we have created a single excitation, but it has emitted a free space photon, represented through $\mathbb{J}_*\rho_m$. As we explained before, after appropriate repumping, the errors from undetected collective quantum jumps can be corrected introducing some extra spontaneous emission probability, such the final density matrix after repumping is given by:

$$\rho_m \to (1-x^2)\rho_m + \left(p_{\text{pump},*} + p_*\right)\mathbb{J}_*\rho_m + O(x^4)\,, \tag{SM16}$$

Fortunately, the errors from spontaneous emission are not so severe as the resulting state still have a big overlap with the original state:

$$\langle\Phi_m|\mathbb{J}_*[\rho_m]|\Phi_m\rangle = \frac{\binom{N-1}{m}}{\binom{N}{m}} \approx 1 - \frac{2m}{N}\,. \tag{SM17}$$

Therefore, the error introduced per failed attempt is given by:

$$\varepsilon_{\text{fail},*} = \left(p_{\text{pump},*} + p\right)\frac{m}{N} \tag{SM18}$$

- *Complete process:* The complete process consists (in average) of a successful heralding event and $1/p$ repetitions such that the average final (in)fidelity to generate $S_{sg}|\Phi_m^{s_1}\rangle$ is given by:

$$I_{m\to m+1} \approx \frac{\varepsilon_{\text{fail},*}}{p} + \varepsilon_{\text{double}} \approx \frac{m}{\eta N P_{1\text{d}}} + (1-\eta)x^2\,, \tag{SM19}$$

Therefore, the best fidelity that can be obtained is done by imposing: $x^2 = \frac{m}{\eta(1-\eta)NP_{1\text{d}}}$, however, at the price of bad scaling of probability: $p = \frac{m}{NP_{1\text{d}}(1-\eta)}$ [2]. The total error to accumulate m excitations depends on how to combine excitations from s_1 and s and will be discussed in Section SM5.

D. Accumulating excitations in the same level.

If we accumulate excitations directly in s, the initial state in each step will be $|\Phi_m^s\rangle$ instead of $|\Phi_m^{s_1}\rangle$. The protocol works in the same way, but the heralding probability has in this case a small correction in m which reads:

$$p_m = p_{\text{coll},m} \times \eta \approx \eta x^2\left(1 - \frac{1}{(m+1)P_{1\text{d}}}\right). \tag{SM20}$$

Moreover, in case of failure we reinitialize the process from the beginning, pumping back all the atoms to g, which avoids the need of applying the repumping protocol discussed below. The infidelity at each step in this case only come from double excitations contributions scale as:

$$I_{m\to m+1} = \varepsilon_{\text{double}} \approx (1-\eta)x^2\,. \tag{SM21}$$

The total error to accumulate m collective excitations will be directly $I_m = m(1-\eta)x^2$.

SM3. HERALDING SINGLE COLLECTIVE EXCITATIONS USING ATOM-WAVEGUIDE QED: TWO-STEP PROTOCOL WITH EXPONENTIAL SCALING.

In this Section we discuss the second protocol of the main manuscript which overcomes the trade-off between heralding probabilities and fidelities. The starting point to analyze this protocol is that the target ensemble is in $|\Phi_m^s\rangle$. Then, the main part of the protocol consist in two steps: i) we use a single atom to provide single excitations to the target ensemble; ii) then, we use an independent ensemble to herald the successful transfer of the excitations to the target ensemble.

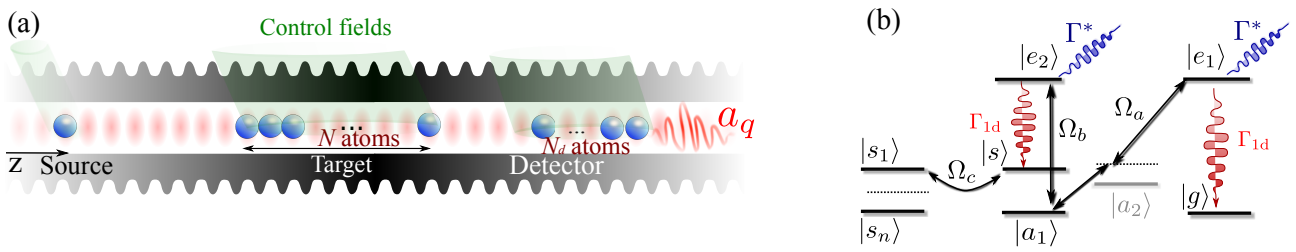

Figure SM2. (a) General setup for the second/third protocol consisting in three individually ensembles: a single *source* atom and a *target/detector* ensembles and N/N_d atoms respectively. (b) Internal level structure of emitters in which the transition $e_1(,e_2) \leftrightarrow g(,s)$ is coupled to a waveguide mode with rate Γ_{1d}. The transition $e_1 \leftrightarrow a_1$ is driven by a two-photon off resonant process such that there is no spontaneous emission associated to it. The transition $e_2 \rightarrow s$ is controlled through a laser with Rabi frequency Ω_b, and the transition $s \leftrightarrow s_n$ is controlled through microwave/Raman laser Ω_c. In the case of the target atoms, we require extra hyperfine levels $\{s_n\}_n$, to store atomic excitations.

A. Atom-waveguide resources.

The requirements that we need for this protocol are:

- We need [see Fig. SM2(a)] one individually addressable *source* atom, and two independently addressable ensembles with N, N_d atoms respectively, one where we store the collective excitations, namely, the *target* ensemble and another one that we use to herald the excitations, namely, the *detector* ensemble. We assume that the atoms have a general level structure as depicted in Fig. SM2(b), with two dipolar transitions $e_1(,e_2) \leftrightarrow g(,s)$ coupled to a waveguide mode with an effective rate Γ_{1d}.

- The atoms are placed again at distances $z_n = n\lambda_a$, such that the dynamics induced by the waveguide is given by

$$\mathscr{L}_{\text{coll}}[\rho] = \frac{\Gamma_{1d}}{2} D_{\sigma^s_{ge_1} + S_{ge_1} + S^d_{ge_1}}[\rho] + \frac{\Gamma_{1d}}{2} D_{\sigma^s_{se_2} + S_{se_2} + S^d_{se_2}}[\rho]. \tag{SM22}$$

  where $S_{\alpha\beta}, S^d_{\alpha\beta}$ denote the collective spin operators in the target/detector atoms.

- The excited states may also decay to other channels with a rate Γ^*. However, to obtain the desired scaling we use a closed transition [3–5] for the $e_1 \rightarrow a_1$, such that the Lindblad term describing the free space spontaneous emission is given by:

$$\mathscr{L}_*(\rho) = \sum_n \left(\frac{\Gamma^*}{4} D_{\sigma^n_{se_2}}[\rho] + \frac{\Gamma^*}{4} D_{\sigma^n_{a_1 e_2}}[\rho] + \frac{\Gamma^*}{2} D_{\sigma^n_{ge_1}}[\rho] \right), \tag{SM23}$$

  where we define the coefficients accompanying the jump operators, such that both transitions have the same P_{1d}.

- Finally, we also require the existence of Raman/microwave fields, $\Omega_{a,b,c}$ as depicted in Fig. SM2(a) that classically control the transition between different atomic levels.

B. Protocol and calculation of probabilities

The protocol for heralding single collective excitation starts with the source/target/detector atoms in $|a_1\rangle_s \otimes |\Phi^s_m\rangle_t \otimes |s\rangle^N_d$ and it consists of several steps (as depicted in Fig. 2 of the main manuscript):

1. First, we move the target/detector atoms in s to an auxiliary level s_1, such that we have $|a_1\rangle_s \otimes |\Phi^{s_1}_m\rangle_t \otimes |s_1\rangle^{N_d}$, avoiding that the excitations from the waveguide can be absorbed the detector level s.

2. Then, we excite the source atom to e_1, then, we switch on Ω^t_a under the Quantum Zeno dynamics conditions ($\Omega^d_b \ll N_d P_{1d}$), such that the excitation is coherently transferred to the ensemble to a_1, i.e., $S_{a_1 g}|\Phi^{s_1}_m\rangle$.

3. Now, we move back the states in the target/detector ensemble from s_1 to s, such that, we prepare $S_{a_1 g}|\Phi^s_m\rangle_t \otimes |s\rangle^{\otimes N_d}_d$. Moreover, we move the source atom to, e.g., s_1, such that it plays no role in subsequent steps.

4. Afterwards, we do a fast π-pulse in the ensemble with Ω_b, such that, $S_{a_1 g}|\Phi^s_m\rangle \rightarrow S_{e_2 g}|\Phi^s_m\rangle$. Afterwards, we apply Ω^d_b within the Zeno dynamics conditions ($\Omega^d_b \ll N_d P_{1d}$), such that we arrive to a state in the target/detector atoms $|\Phi^s_{m+1}\rangle \otimes S_{a_1 s}|s\rangle^{\otimes N_d}$.

5. Thus, if we measure the detector atoms in a_1, we herald the transfer of a single collective excitation to s.

Assuming that π-pulses are perfect, the relevant steps of the protocols for the analysis of probabilities and fidelities are the second and the fourth. Interestingly, they can be analyzed in a common way as in each of them what happens is that a single excitation is transferred through the waveguide via Zeno dynamics to an ensemble with N/N_d atoms respectively. Therefore, in the following Section we analyze the general problem and then we will particularize for steps 2 and 4.

1. *General Zeno step*

The general problem consists of two ensembles (a and b) with three-level atoms (with metastable states $|0\rangle$, $|1\rangle$ and excited state $|2\rangle$) that contain N_a and N_b atoms each. The dynamics are governed by the collective decay on the $|2\rangle-|1\rangle$ transition, i.e., $\mathscr{L}[\rho]=\frac{1}{2}\Gamma_{1\mathrm{d}}D_{S_{12}^{(\mathrm{a})}+S_{12}^{(\mathrm{b})}}[\rho]$ and an external field on the second ensemble, i.e., $H_\mathrm{b}=\frac{1}{2}\Omega S_{21}^{(\mathrm{b})}+\mathrm{h.c.}$. Therefore the effective non-hermitian hamiltonian that drives the no-jump evolution is given by:

$$H_\mathrm{eff}=\frac{1}{2}\left(\Omega S_{21}^{(\mathrm{b})}+\mathrm{h.c.}\right)-\mathrm{i}\frac{\Gamma_{1\mathrm{d}}}{2}\left(S_{21}^{(\mathrm{a})}+S_{21}^{(\mathrm{b})}\right)\left(S_{12}^{(\mathrm{a})}+S_{12}^{(\mathrm{b})}\right)-\mathrm{i}\frac{\Gamma^*}{2}\sum_n\sigma_{ee}^n. \tag{SM24}$$

Denoting the collective symmetric excitations as $|\#_0,\#_1,\#_2\rangle_{\mathrm{a/b}}$, the initial state of the problem we want to analyze can be written $|0,0,1\rangle_\mathrm{a}\otimes|0,N-m,0\rangle_\mathrm{b}$ and the goal state is $|0,1,0\rangle_\mathrm{a}\otimes|1,N-m-1,0\rangle_\mathrm{b}$. The decay operators couple an initial state $|\psi_1\rangle=|N_\mathrm{a}-k-1,k,1\rangle_\mathrm{a}\otimes|0,N_b,0\rangle$ to $|\psi_2\rangle=|N_\mathrm{a}-k-1,k+1,0\rangle_\mathrm{a}\otimes|0,N_b,1\rangle$ and the coherent terms couple the latter to $|\psi_3\rangle=|N_\mathrm{a}-k-1,k+1,0\rangle_\mathrm{a}\otimes|1,N_b,0\rangle$. In the basis of superradiant and dark states, that is $\{|\psi_s\rangle,|\psi_d\rangle,|\psi_3\rangle\}$ with $|\psi_s\rangle=\sqrt{\frac{k+1}{N_\mathrm{b}+k+1}}|\psi_1\rangle+\sqrt{\frac{N_\mathrm{b}}{N_\mathrm{b}+k+1}}|\psi_2\rangle$ and $|\psi_d\rangle=\sqrt{\frac{N_b}{N_\mathrm{b}+k+1}}|\psi_1\rangle-\sqrt{\frac{m+1}{N_\mathrm{b}+k+1}}|\psi_2\rangle$, and assuming the coherent driving is weak compared to the collective dissipation *Omega* $\ll N_\mathrm{b}\Gamma_{1\mathrm{d}}$, the superradiant state can be adiabatically eliminated and an effective Zeno Dynamics between the dark states is obtained. To minimize the errors, one chooses $\Omega=\sqrt{(N_\mathrm{b}+k+1)\Gamma_{1\mathrm{d}}\Gamma^*}$ such that $\frac{N_\mathrm{b}|\Omega|^2}{(N_\mathrm{b}+k+1)^2\Gamma_{1\mathrm{d}}}=\Gamma^*$. The dynamics for each state is then approximately given by

$$|\psi_d(t)|^2\approx\frac{N_\mathrm{b}}{N_\mathrm{b}+k+1}e^{-\Gamma^*t}\cos^2\left(t\frac{\Gamma^*\sqrt{(k+1)P_{1\mathrm{d}}}}{2}\right), \tag{SM25}$$

$$|\psi_3(t)|^2\approx\frac{N_\mathrm{b}}{N_\mathrm{b}+k+1}e^{-\Gamma^*t}\sin^2\left(t\frac{\Gamma^*\sqrt{(k+1)P_{1\mathrm{d}}}}{2}\right)\equiv p, \tag{SM26}$$

$$|\psi_s(t)|^2\approx\frac{k+1}{N_\mathrm{b}+k+1}e^{-(\Gamma^*+(N_\mathrm{b}+k+1)\Gamma_{1\mathrm{d}})t}. \tag{SM27}$$

The success probability $p=|\psi_3(T)|^2$ is the maximized for $T=\pi\left(\sqrt{\frac{k+1}{N_\mathrm{b}+k+1}}\Omega\right)^{-1}$ and $\Omega=\sqrt{(N_\mathrm{b}+k+1)\Gamma_{1\mathrm{d}}\Gamma^*}$ for $k\ll N_\mathrm{b}$. In this case, one obtains

$$p=\frac{N_\mathrm{b}}{N_\mathrm{b}+k+1}\mathrm{e}^{-\pi/\sqrt{(k+1)P_{1\mathrm{d}}}}, \tag{SM28}$$

where the prefactor originates in the non-unit overlap of the initial state with the dark state, i.e., $|\langle\psi_d|\psi_1\rangle|^2=\frac{N_\mathrm{b}}{N_\mathrm{b}+k+1}$.

For the repumping protocols that we study in Section SM4 it is important to know the probability of spontaneous jumps in both ensembles during the evolution. As we already saw in the first protocol, the problematic processes are the one associated to leaky photons. The quantum jump analysis shows that the probability for a spontaneous jump in the ensemble a or b is given by

$$p_{\mathrm{a},*}=p_{\mathrm{a}_1,*}+p_{\mathrm{a}_2,*}=\Gamma^*\int_0^T\mathrm{d}t_1|\psi_1(t_1)|^2+\Gamma^*\int_0^\infty\mathrm{d}t_1|\psi_1(T)|^2e^{-\Gamma^*t_1}, \tag{SM29}$$

$$p_{\mathrm{b},*}=p_{\mathrm{b}_1,*}+p_{\mathrm{b}_2,*}=\Gamma^*\int_0^T\mathrm{d}t_1|\psi_2(t_1)|^2+\Gamma^*\int_0^\infty\mathrm{d}t_1|\psi_2(T)|^2e^{-\Gamma^*t_1}, \tag{SM30}$$

where the first parts, $p_{\mathrm{a,b}_1,*}$, corresponds to the interval of time $(0,T)$ where Ω is switched on, and the second part, $p_{\mathrm{a}_2,*}$, the time $t\gg1/\Gamma^*$ that we wait, with $\Omega=0$, such that all the population in the excited state, if any, disappears. By using the approximations for H_eff, we can calculate the different contributions and upper bound the probabilities from these processes

$$p_{\mathrm{a}_1,*}\lesssim\frac{1}{2}(1-e^{-\pi/\sqrt{(k+1)P_{1\mathrm{d}}}})\lesssim\frac{\pi}{2\sqrt{P_{1\mathrm{d}}}} \tag{SM31}$$

$$p_{\mathrm{b}_1,*}\lesssim\frac{k+1}{2N_\mathrm{b}}(1-e^{-\pi/\sqrt{(k+1)P_{1\mathrm{d}}}})\lesssim\frac{\pi\sqrt{k+1}}{2N_\mathrm{b}\sqrt{P_{1\mathrm{d}}}}, \tag{SM32}$$

which mainly comes from the contribution of the dark state and where the last approximation is valid for $P_{1d} \gg 1$.

Finally, we need to consider what happens with the contribution $p_{a_2,*}$ when $P_{1d} \gg 1$, and when we assume a perfect timing, $T = \pi/\sqrt{(k+1)\Gamma_{1d}\Gamma^*}$ such that then $|\psi_d(T)|^2 = 0$. The only contribution remaining is the one of the superradiant $|\psi_s(T)\rangle = \sqrt{\frac{k+1}{N_b+k+1}} e^{-(\Gamma^*+(N_b+k+1)\Gamma_{1d})T/2}$, which leads to

$$p_{a_2,*} \lesssim \frac{(k+1)^2}{(N_b+k+1)^2} e^{-\pi(N_b+k+1)/\sqrt{(k+1)P_{1d}}}, \tag{SM33}$$

$$p_{b_2,*} \lesssim \frac{N_b(k+1)}{(N_b+k+1)^2} e^{-\pi(N_b+k+1)/\sqrt{(k+1)P_{1d}}}, \tag{SM34}$$

which are negligible compared to $p_{a,b_1,*}$ for sufficiently large N_b.

2. *Particularizing for the two step protocol: success probability and infidelity.*

The second step of the protocol that we want to analyze corresponds to identifying in the general problem $|0\rangle = |a_1\rangle$, $|1\rangle = |g\rangle$, $|2\rangle = |e_1\rangle$, $\Omega = \Omega_a$, where the ensembles have a size of $N_a = 1$ and an effective size $N_b = N - m$ because the m excitations in all other states are decoupled and effectively reduce the atom number. This results in a heralding probability:

$$p_a = \frac{N-m}{N-m+1} e^{-\pi/\sqrt{P_{1d}}}. \tag{SM35}$$

In the fourth step, the equivalence reads $|0\rangle = |a_1\rangle$, $|1\rangle = |s\rangle$, $|2\rangle = |e_2\rangle$, $\Omega = \Omega_b$ and the ensembles have a size of $N_a = N$ and $N_b = N_d$. The initial state is $|N-m-1,m,1\rangle_a \otimes |0,N_d,0\rangle_b$ and the goal state is $|N-m-1,m+1,0\rangle_a \otimes |1,N_d-1,0\rangle_b$. Thus, the success probability reads:

$$p_b = \frac{N_d}{N_d+m+1} e^{-\pi/\sqrt{(m+1)P_{1d}}}, \tag{SM36}$$

where it can be shown that in the limit of $N, N_d \gg m$, the total probability of success is lower bounded by: $p \gtrsim p_a^2 \approx e^{-2\pi/\sqrt{P_{1d}}}$.

SM4. HERALDING SINGLE COLLECTIVE EXCITATIONS USING ATOM-WAVEGUIDE QED: TWO-STEP PROTOCOL FOR POLYNOMIAL SCALING.

In this Section we analyze the third protocol discussed in the manuscript. This protocol is a variation of the second protocol in which after each successful heralding we store the excitations in different levels $\{s_n\}_n$ to combine them a posteriori. This however implies the modification of the previous protocol to obtain the desired $1/(NP_{1d})$ scaling for the infidelity $I_{m\to m+1}$. In particular, the second step can not be done through Quantum Zeno dynamics, because the probability of emitting an spontaneous photon within the target ensemble is large, i.e., $p_{b,*} \propto 1/\sqrt{P_{1d}}$, that would yield an infidelity $I_{m\to m+1} \propto 1/(N\sqrt{P_{1d}})$. Therefore, in order to achieve subexponential scaling in the number of operations *and* infidelities still scaling with $I_m \propto 1/(NP_{1d})$ one needs to make judicious modification of the setup as we will explain now.

A. Atom-waveguide resources for two-step protocol.

The requirements of this protocol are exactly analogue to the ones of Section SM3 with two modifications: i) we only require a single detector atom, i.e., $N_d = 1$[6]; ii) we require the existence of several hyperfine states $\{s_n\}$ to store the superpositions.

B. Protocol and calculation of probabilities

The initial state of the target ensemble for this protocol is given by:

$$|\Phi_m^{\{s_n\}}\rangle = \frac{1}{\sqrt{\binom{N}{m_1,m_2,\ldots,m_n}}} \mathrm{sym}\{|s_1\rangle^{\otimes m_1} \ldots |s_n\rangle^{\otimes m_n} |g\rangle^{\otimes N_m}\} \tag{SM37}$$

where $\binom{N}{m_1,m_2,\ldots,m_n} = \frac{N!}{m_1!\ldots m_N N_m)!}$ is the number of states within the superposition, denoting $N_m = N - m$. The final goal state we want to create is:

$$|\Phi_{\text{goal}}\rangle = \frac{1}{\sqrt{N_m}} S_{sg} |\Phi_m^{\{s_n\}}\rangle \,. \tag{SM38}$$

As we mention, the collective transfer of the excitation from the source to the target ensemble is analogous than the one discussed in Section SM3. Thus, we focus only the second step which should be done different.

1. *Step (b): heralding the transfer using fast π-pulses (no-jump evolution).*

This second step of the protocol do not rely on quantum Zeno dynamics, and it will take place between the target and detector atoms. The idea is to: i) first do a fast π pulse with $\Omega_b^{\text{t}} \gg \Gamma_{1\text{d}}$ such that the possible excitation in $S_{a_1 g}|\Phi_m^{\{s_n\}}\rangle \to S_{e_2 g}|\Phi_m^{\{s_n\}}\rangle$; ii) let the system evolve only through the couplings induced by the waveguide; iii) do a π pulse in both the target and detector ensemble, i.e., $\Omega_b^{\text{t}}, \Omega_b^{\text{d}} \gg \Gamma_{1\text{d}}$, such that any remaining excitation in the excited states e_2 go back to a_1. It is important to notice that if no excitation has been transferred to a_1 in step (a), nothing will happen in this step. Therefore, we consider our initial state to be: $|\Psi_{0,\text{b}}\rangle = |\Psi_{b,1}\rangle = \frac{1}{\sqrt{N_m}} S_{e_2 g}|\Phi_m^{\{s_n\}}\rangle \otimes |s\rangle$. The effective Hamiltonian in this case is then given by:

$$H_{\text{b,eff}} = -i\frac{\Gamma^*}{2}\sum_n \sigma^n_{e_2 e_2} - i\frac{\Gamma_{1\text{d}}}{2}\left(S_{e_2 s} + \sigma^{\text{d}}_{e_2 s}\right)\left(S_{s e_2} + \sigma^{\text{d}}_{s e_2}\right). \tag{SM39}$$

From here, it is simple to arrive to the solution of the dynamics: $|\Psi_{\text{b}}(t)\rangle = e^{-iH_{\text{b,eff}}t}|\Psi_{0,\text{b}}\rangle = \sum_j \beta_j(t)|\Psi_{\text{b},j}\rangle$, which read:

$$|\beta_1(t)|^2 = \frac{1}{4}e^{-\Gamma^* t}\left[1 + e^{-\Gamma_{1\text{d}} t}\right]^2 \tag{SM40}$$

$$|\beta_2(t)|^2 = \frac{1}{4}e^{-\Gamma^* t}\left[1 - e^{-\Gamma_{1\text{d}} t}\right]^2 \equiv p_{\text{b}}\,, \tag{SM41}$$

Notice that p_{b} will be the probability of having transferred the collective excitation to the target while changing the detector atom state to a_1, if we assume that the second π pulse is perfect. If we choose a time $T_{\text{b}} = 1/\Gamma_{1\text{d}}$, then

$$|\beta_1(T_{\text{b}})|^2 \approx \frac{(e+1)^2}{4e^2}\left(1 - \frac{1}{P_{1\text{d}}}\right) = 0.46\left(1 - \frac{1}{P_{1\text{d}}}\right), \tag{SM42}$$

$$p_{\text{b}} \approx \frac{(e-1)^2}{4e^2}\left(1 - \frac{1}{P_{1\text{d}}}\right) \approx 0.1\left(1 - \frac{1}{P_{1\text{d}}}\right), \tag{SM43}$$

Interestingly, there is a sizeable probability of remaining in the initial state of this step. So instead of start from the beginning every time that we fail, it is possible to repeat this step several times, which increases probability to $p_b \approx 1/3$.

2. *Step b: heralding the transfer using fast π-pulses (quantum jump analysis).*

The relevant quantum jumps in this step are $J_{\text{b}}\rho = J_{\text{b},*}\rho + J_{\text{b,coll}}\rho$, with

$$\begin{aligned} J_{\text{b},*}\rho &= \frac{\Gamma^*}{2}\sum_n \sigma^n_{s e_2}\rho\,\sigma^n_{e_2 s} + \frac{\Gamma^*}{2}\sum_n \sigma^n_{a_1 e_2}\rho\,\sigma^n_{e_2 a_1}\,, \\ J_{\text{b,coll}}\rho &= \Gamma_{1\text{d}}\left(S_{s e_2} + \sigma^{\text{d}}_{s e_2}\right)\rho\left(S_{e_2 s} + \sigma^{\text{d}}_{e_2 s}\right). \end{aligned} \tag{SM44}$$

In this second step, we make a π-pulse with Ω_b^{t}, such that the initial state $|\Psi_{\text{b},0}\rangle = |\Psi_{\text{b},1}\rangle = \frac{1}{\sqrt{N_m}} S_{e_2 g}|\Phi_m^{\{s_n\}_n}\rangle$. Then the system is left free to evolve for a time $T_{\text{b}} = \frac{1}{\Gamma_{1\text{d}}}$, and switching the $\Omega_b^{\text{t}}, \Omega_b^{\text{d}}$ such that no population remains in the excited state e_2. Therefore, the quantum jumps could only occur within a small time interval giving rise to

$$p_{\text{b},*} = \Gamma^* \int_0^{T_{\text{b}}} dt_1 |\beta_1|^2 \approx \frac{0.67}{P_{1\text{d}}}\,. \tag{SM45}$$

In this regime, with $P_{1\text{d}} \gg 1$ and $T_{\text{b}} = \frac{1}{\Gamma_{1\text{d}}}$, it is also instructive to consider the probability of other quantum jumps. For example, to calculate the probability that the excitation is transferred to $S_{sg}|\Phi_m^{\{s_n\}}\rangle$ but the state of the detector is unchanged, i.e.,

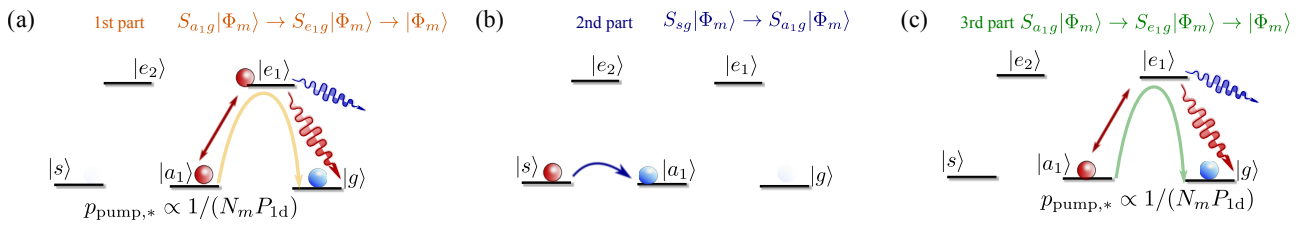

Figure SM3. General scheme of repumping process to correct collective quantum jump errors (see details in the text).

remains in s, one needs to sum up the probability of making a collective quantum jump in the target together with the probability of making any jump in the $f \to s$ transition:

$$p_{\text{b,coll}} = \Gamma_{1\text{d}} \int_0^{T_\text{b}} dt_1 |\beta_1|^2 + (\Gamma_{1\text{d}} + \frac{\Gamma^*}{2}) \int_0^{T_\text{b}} dt_1 |\beta_2|^2 \approx 0.71\,, \tag{SM46}$$

which biggest contribution comes from the emission in the target ensemble. These states have not destroyed the coherence of $|\Phi_m^{\{s_n\}}\rangle$ but they need to be properly taken care of with the repumping process. In order to avoid that these states also lead to error, we have to make a repumping process back to g in the way depicted in Fig. SM3:

1. First we pump incoherently back any possible excitation in c through the waveguide, namely, $S_{a_1g}|\Phi_m^{\{s_n\}}\rangle \to S_{e_1g}|\Phi_m^{\{s_n\}}\rangle \to |\Phi_m^{\{s_n\}}\rangle$. Because this process is done through a collective photon the error introduced in this step $p_{\text{pump},*} \propto \frac{1}{NP_{1\text{d}}}$.

2. Once, we have make sure that there are no excitation in a_1 we can move the excitations from $S_{sg}|\Phi_m^{\{s_n\}}\rangle \to S_{a_1g}|\Phi_m^{\{s_n\}}\rangle$ with π pulse using a microwave or two-photon Raman transition.

3. Now, after having transferred the excitation to a_1, we repeat the incoherent transfer through the waveguide that will only introduce errors of the order $p_{\text{pump},*}$.

C. Fidelities of the protocol.

From the calculation of probabilities of the previous Section, we are prepared to estimate the average fidelity when detecting an atomic excitation in the detector atom in a_1. The errors appear:

- *Successful heralding:* As shown in the main manuscript, in the case where the transition $e_1 \to a_1$ is closed the only way of measuring detector atom in a_1 is by successfully generating $|\Phi_{\text{goal}}\rangle$. If we have a spurious spontaneous emission $\sim \alpha\Gamma^*$, then, it is possible that we herald the transfer of an incoherent excitation of s, i.e., σ_{sg}^n, which have an overlap $\propto 1/N$ with the state we want to create. Thus, the possible error after heralding is given by:

$$\varepsilon_{\text{closed}} = \frac{\alpha}{N\sqrt{P_{1\text{d}}}}\,. \tag{SM47}$$

- *Failed heralding:* As it occurred in the first protocol, in each failed attempt we introduce an error in our initial state $|\Phi_m^{\{s_n\}}\rangle$ due to the probability of emitting free space, which in this protocol may occur in step (a), (b) and the repumping. The error per attempt can be estimated to be:

$$\varepsilon_* \approx (p_{\text{a},*} + p_{\text{b},*} + p_{\text{pump},*})\frac{m}{N} \approx \frac{0.67m}{NP_{1\text{d}}}\,. \tag{SM48}$$

  where in the last approximation we assumed to be in a regime where $p_{\text{b},*} \gg p_{\text{a},*}, p_{\text{pump},*}$.

- *Complete process:* As in the first protocol, the average complete process consist of a successful heralding and $1/p$ failed attempts, such that the average fidelities of the process:

$$I_{m\to m+1} = \varepsilon_{\text{closed}} + \frac{\varepsilon_*}{p} = \frac{\alpha}{N\sqrt{P_{1\text{d}}}} + \frac{m}{pNP_{1\text{d}}}\,, \tag{SM49}$$

  where one sees that to reach the $I_{m->m+1} \propto 1/(NP_{1\text{d}})$ scaling, we do not need a perfectly closed transition, but it is enough to demand a cancellation of spontaneous emission (α) of at least $\alpha = 1/\sqrt{P_{1\text{d}}}$ which can be achieved as well, e.g., by using a quadrupole transition.

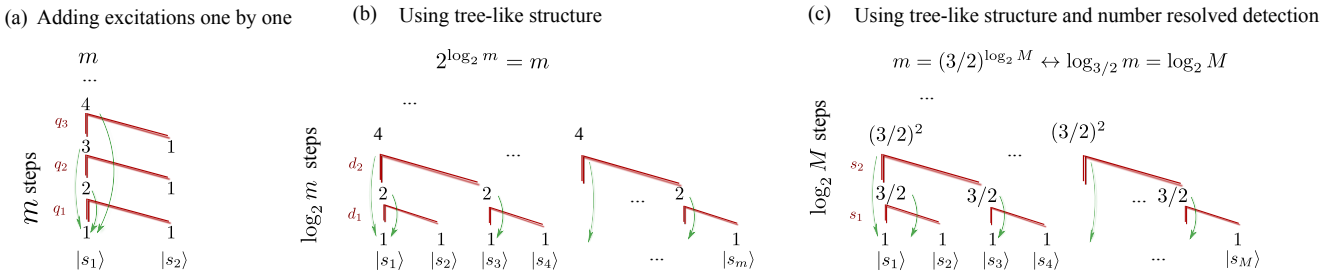

Figure SM4. Scheme to build up m collective atomic states combining heralded single collective excitations plus beam splitters and postselection. The red arrows indicates the path adding excitations, whereas the green ones how one needs to reinitialize the process if we fail in given step. (a) Scheme using two atomic levels to add excitations one by one with probability q_n in each step. (b) [(c)] Scheme using $\log_2 m$ [$\log_{3/2} m$] with probability q_n [s_n] in each step using a tree-like structure that avoids starting the whole process from the beginning in case of failure.

SM5. MERGING PROTOCOLS FOR COMBINING COLLECTIVE ATOMIC EXCITATIONS.

In this Section we show how to combine the excitations stored in the target atoms after the first part of the process, that is, when our atomic state has the form $|\Phi_m^{\{s_n\}}\rangle \propto \mathrm{sym}\{|s_1\rangle^{m_1} \otimes \cdots \otimes |s_n\rangle^{m_n} \otimes |g\rangle^{N_m}\}$, with $N_m = N - m = N - \sum_i m_i$ such that we can build up higher excitation numbers into a single level. For that purposes we use several tools and assumptions:

- We are interested in the Holstein-Primakoff limit where $m \ll N$, such that $S_{\alpha g}\sqrt{N_m} \approx a_\alpha^\dagger$, with $[a_\alpha, a_\alpha^\dagger] = 1$
- Using the microwave/Raman lasers connecting an atomic level α with g, we can do displacement transformations $D(\alpha)$ on the α level. Moreover, using microwave/Raman transition between α and β levels, we can also engineer beam splitter transformations between these modes.
- Moreover, we can read the atomic state very efficiently by pumping to an excited state that emits a collective photon through the waveguide in a cyclic transition.

Using all those assumptions, our goal is to build a given arbitrary atomic state of m excitations, that we know it can be afterwards be mapped to a photonic state with very high efficiency [7]. The general form of this state typically reads:

$$|\Psi_m\rangle = \sum_n \frac{f_n}{\sqrt{n!}}(a^\dagger)^n|\mathrm{vac}\rangle \propto \prod_{i=1}^{n}(a^\dagger - \alpha_i^*)|\mathrm{vac}\rangle\,, \tag{SM50}$$

and we will put particular emphasis in the simple Dicke states with m excitations, as this will be mapped to photonic Fock states. We discuss two different types of merging protocols separately: i) If the number of metastable states s_i is limited, we only use two levels and add excitations one by one; ii) Using several hyperfine levels and adopting a tree-like structure that will allow us to avoid the exponential scaling of the number of operations.

A. Adding excitations one by one.

The general scheme to build a given atomic state with m collective excitations is depicted in Fig. SM4(a). The main step consists of going from an state with m excitations in one level and a single one in the other, $|\phi_n\rangle = |n\rangle \otimes |1\rangle$, to an state $n+1$ excitation in one of them conditioned on detecting no excitation in the other one, i.e., $|n+1\rangle \otimes |0\rangle_1$. The idea is to apply a beam splitter transformation of $|\phi_n\rangle$, followed by a projection of the state conditioned on detecting no excitations in the a_2 level:

$$|\tilde{\phi}_n\rangle = \mathbb{P}_{0_2} B|\phi_n\rangle = \frac{1}{\sqrt{n!}} T^n R (a_1^\dagger)^{n+1}|\mathrm{vac}\rangle \tag{SM51}$$

(not normalized) and the probability of finding it is given by

$$\langle\tilde{\phi}_n|\tilde{\phi}_n\rangle = (n+1)|T|^{2n}|R|^2 \leq \left(\frac{n}{n+1}\right)^m = q_n, \tag{SM52}$$

which is maximized for a transmitivity $|T|^2 = n/(n+1)$. Interestingly, $\lim_{n\to\infty} q_n = e^{-1}$ such that it does not decay with n. However, if we fail, we have to repeat the process from the beginning, such that the mean number of operations to create a state

with m excitations can be obtained:

$$R_m = \frac{1+R_{m-1}}{q_{m-1}} = \frac{1}{q_{m-1}} + \frac{1}{q_{m-1}q_{m-2}} + \cdots + \frac{1}{\prod_n^{m-1} q_n}. \tag{SM53}$$

Taking into account that $2 = q_1 > q_2 > \cdots > q_n > e^{-1}$ and $q_0 = p$ the probability to generate a single heralded excitation, the mean number of states can be lower and upper bounded by:

$$\frac{2^{m-1}}{p} < R_m < \frac{me^m}{p}, \tag{SM54}$$

which increase exponentially to increase the number of excitations m. Moreover, it was shown in Ref. [8] that by combining single photon addition with displacement transformation one can obtain any arbitrary superposition as in Eq. SM50, but also with an exponential number of operations.

B. Doubling the number of excitations post-selecting on no detection.

The key difference with respect to the previous protocol is that we are going to adopt a tree-like structures as depicted in Fig. SM4(b), where we can reach the m excitations in $\log_2 m$ steps and we do not have to repeat all of them if we fail. The first building block is to study the process that add up two states with $|n\rangle$ excitations to go to $|n\rangle \otimes |n\rangle \to |2n,0\rangle$. It can be shown that the conditional outcome after a beam splitter and detecting no excitations in the second atomic level is given by:

$$|\tilde{\phi}_{2n}\rangle = \frac{1}{n!} T^n R^n (a_1^\dagger)^{2n} |\text{vac}\rangle, \tag{SM55}$$

which yields an (approximated) optimal probability for a $50-50$ beam splitter transformation:

$$d_n = \langle \tilde{\phi}_{2n} | \tilde{\phi}_{2n} \rangle = \frac{(2n)!}{2^{2n}(n!)^2} \approx 1/\sqrt{\pi n}, \tag{SM56}$$

However, in spite of this decay, the use of of a tree-like structure is enough to circumvent the exponential scaling of adding excitations one by one. The average number of operations to arrive to an state of m excitations can be calculated iteratively, i.e., $R_m = d_{m/2}^{-1}$, where the $d_{m/2}^{-1}$ terms represents the number of times we need to try in the step $m/2$ to succeed and the one with $2R_{m/2}$ is the number of steps we need to repeat at the m-th step to get the the two branches of the tree. We lower and upper bound this number of operations to get m excitations by:

$$\frac{m^2}{4p} < R_m < \frac{m^{(\log_2 m)/2+1} \log_2 m}{2p} \tag{SM57}$$

Thus, R_m is superpolynomial in m because the probability in each step, d_m, decays with m. For completeness, let us mention that it is also possible to prove that one can also get arbitrary superpositions [9] using the tree-like structure of Fig. SM4(b).

C. Increasing the number of excitations with number resolved detection.

In this last Section, we show how by using atomic number resolved detection, we can overcome the superpolynomial scaling of the previous section. Instead of starting the process again when we detect some excitation in the a_2, we can think of using some of these states that still have a non-negligible number of excitations in the a_1 mode that we can use a posteriori. To further explore this possibility, we generalize the operation of Eq. SM55 to see what is the resulting state after $50/50$ beam splitter transformation when we want to sum up m and n excitations in the $a_{1,2}$ modes when we detect the p excitations in the a_2 mode:

$$\begin{aligned} f_p(m,n) &=_1 \langle m+n-p|\tilde{\phi}_{m+n-p}\rangle =_1 \langle m+n-p|\mathbb{P}_{p_2} B \frac{(a_1^\dagger)^m (a_2^\dagger)^n}{\sqrt{m!}\sqrt{n!}}|\text{vac}\rangle \\ &= \sum_{k=0}^{m} \frac{(-1)^{m-k}}{2^{(m+n)/2}\sqrt{m!}\sqrt{n!}} \binom{m}{k} \binom{n}{m+n-k-p} \sqrt{p!}\sqrt{(m+n-p)!}. \end{aligned} \tag{SM58}$$

For the symmetric situation the expression of Eq. SM58 can be simplified as follows:

$$|f_{2n}(m,m)|^2 = \frac{(2m-2n)!(2n)!}{2^{2m}(m!)^2} \binom{m}{n}^2 \approx \frac{1}{\pi} \frac{1}{\sqrt{n(m-n)}}. \tag{SM59}$$

and $f_{2n-1}(m,m) \equiv 0$ for $n \in \mathbb{N}$, and where in the last approximation we assumed $m,n \gg 1$ to use Stirling approximation. By integrating this expression it can be obtained that the probability of detecting an state $p < m/2$ lead to:

$$s_{p<m/2} = \sum_{p=0}^{m/2} |f_p(m,m)|^2 \approx \int_0^{m/4} dn |f_{2n}(m,m)|^2 = \int_0^{m/4} dn \frac{1}{\pi} \frac{1}{\sqrt{n(m-n)}} = 1/3 \equiv s\,, \tag{SM60}$$

which means that there is a probability s independent of m of going from $|m\rangle_1 \otimes |m\rangle_2 \to |2m-p > 3m/2\rangle_1 \otimes |p\rangle_2$. In order to make a worst case estimation, we assume that in each step we go only to $|m\rangle_1 \otimes |m\rangle_2 \to |3m/2\rangle_1$. This in principle can be done because if one obtains a higher excitation number it can be decreased by applying infinitesimal beam splitter transformations with an empty mode as:

$$\begin{aligned} \mathbb{P}_{0_2} B(\theta \ll 1) |n\rangle_1 \otimes |0\rangle_2 &\approx \left(1 - \frac{\theta^2 n}{2}\right) |n\rangle\,, \\ \mathbb{P}_{0_2} B(\theta \ll 1) |n\rangle_1 \otimes |0\rangle_2 &\approx i\theta\sqrt{n} |n-1\rangle_1\,, \end{aligned} \tag{SM61}$$

which show that by switching $\theta^2 n \ll 1$, we do not alter $|n\rangle_1$, until we decrease one excitation. This can be applied several times until we arrive to $|3m/2\rangle$. In this worst case scenario, starting again with M single excitations [see Fig. SM4(c)] we will arrive at least to $m \geq \left(\frac{3}{2}\right)^{\log_2(M)}$ [which lead $\log_{3/2} m = \log_2 M$] in $\log_2(M)$ steps, which means that with this protocol

$$R_m \lesssim \frac{\log_2(M) 2^{\log_2 M - 1}}{p s^{\log_2 M}} \approx \frac{m^{4.41} \log_{3/2} m}{2p}\,, \tag{SM62}$$

where we used $\log_b x = \frac{\log_a x}{\log_a b}$. Thus, by using number resolved detection we turn the superpolynomial scaling into polynomial which is big improvement if we want to scale our protocol for larger excitation numbers.

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